

OPERATORS ON BOURGAIN-DELBAEN'S SPACES.

DANIELE PUGLISI

ABSTRACT. We prove that whenever we pick real numbers a, b such that $0 < b < a < 1$, $a + b > 1$ and $a^3 + b^3 = 1$, then every operator from ℓ_2 to $X_{a,b}$ and from $X_{a,b}$ to ℓ_2 must be compact, where $X_{a,b}$ is the Bourgain-Delbaen's space.

1. INTRODUCTION

This note is concerning a question raised to us by R. Aron during the conference "Function Theory on Infinite Dimensional Spaces XIV. 08 - 11 February 2016, Madrid. We were told that the same question was raised during some other conference in 1987 and it was solved by J. Bourgain soon after. Since at that time J. Bourgain decided to not published it, the result remains only announced in [1].

Throughout this note we shall use a standard notation. For Banach spaces X, Y we denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators from X to Y and by $\mathcal{K}(X, Y)$ its ideal of compact operators. It is a classical Pitt's result [4] that every linear continuous operator from ℓ_p to ℓ_q must be compact for all $q < p$. It is worth to say here that this result has been generalized in [5, Lemma 3.5] where ℓ_p -direct sum of sequences of suitable Banach spaces have been used. Following Pitt's theorem, in [1, Lemma 5] it was proved the following

Proposition 1.1. Suppose a Banach space X has the property that for some $p > 1$, $\mathcal{L}(X, \ell_p) = \mathcal{K}(X, \ell_p)$. Then $\mathcal{L}(X, \ell_q) = \mathcal{K}(X, \ell_q)$ for all $1 < q < p$.

This proposition shows that there are non-trivial examples of triple (X, Y, Z) of Banach spaces with the property that if every continuous linear operator from X to Y is compact and every continuous linear operator from Y to Z is compact, then every continuous linear operator from X to Z is compact. In case this happens, we say that the triple (X, Y, Z) has the *compact operators transitivity property*. Therefore, it is quite natural to raise the following

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Question (R. Aron, 1987). Does every triple (X, Y, Z) of Banach spaces has the compact operators transitivity property?

As announced in [1], the answer was given in negative by J. Bourgain and in this note we propose a proof of such result. Another negative solution of the above question was given in [2], where, if we denote by $\mathfrak{X}_{AH}(L)$ the Argyros-Haydon space constructed on the infinite subset $L \subseteq \mathbb{N}$, the following spectacular result holds

Theorem 1.2. If $L \cap L'$ is finite, then every bounded linear operator from $\mathfrak{X}_{AH}(L)$ to $\mathfrak{X}_{AH}(L')$ must be compact.

2. THE BOURGAIN-DELBAEN'S SPACE

In this section we would like to review briefly the construction of the classical Bourgain-Delbaen's space $X_{a,b}$ using the original notation (see [3]), even if we recommend the reader the Ph.D. Thesis of M. Tarbard [6], where the exposition is very clear.

For a $\lambda > 1$ let us fix from now on real numbers a, b such that

- (1) $0 < b < a < 1$,
- (2) $a + 2b\lambda \leq \lambda$,
- (3) $a + b > 1$.

By induction, first we construct a sequence $(d_n)_n \subseteq \mathbb{R}$ and

$$i_{m,l} : \text{span}\{e_i : 1 \leq i \leq d_m\} \longrightarrow \text{span}\{e_i : 1 \leq i \leq d_l\}$$

with $m < l \leq n$, where e_i is the standard basic sequence, i.e. $e_i(j) = \delta_{i,j}$, as follows.

Let $d_1 = 1$. Suppose d_1, \dots, d_n are known as well as $\{i_{m,l} : 1 \leq m < l \leq n\}$ satisfying

- (a) $\pi_m \circ i_{m,n} = \text{id}_{\text{span}\{e_i : 1 \leq i \leq d_m\}}$ for $m < n$,
- (b) $i_{l,n} \circ i_{m,l} = i_{m,n}$, for $m < l < n$,

where π_m is the natural projection to the first d_m coordinates. For all

$$\begin{aligned} \gamma &= (m, i, j, \varepsilon', \varepsilon''), \text{ with } m < n, \\ 1 &\leq i < d_m, \ 1 \leq j \leq d_n \text{ and } \varepsilon', \varepsilon'' = \pm 1 \end{aligned}$$

let us define a functional in $(\text{span}\{e_i : 1 \leq i \leq d_n\})^*$ as

$$c_\gamma^*(x) = \varepsilon' a x_i + \varepsilon'' b (x - \pi_m \circ i_{m,n}(x))_j.$$

Let us consider the set of all such functionals, namely

$$\mathcal{F}_n = \{c_\gamma^* : \gamma = (m, i, j, \varepsilon', \varepsilon''), \ m < n, \ 1 \leq i < d_m, \ 1 \leq j \leq d_n \text{ and } \varepsilon', \varepsilon'' = \pm 1\};$$

then, we define

$$d_{n+1} = d_n + \text{card}\mathcal{F}_n$$

and

$$i_{n,n+1}(x) := \underbrace{(x_1, \dots, x_{d_n}, (c_\gamma^*(x))_{\gamma \in \mathcal{F}_n})}_{d_{n+1}}, 0, 0, \dots, 0, \dots).$$

For each $n \in \mathbb{N}$ let us consider

$$i_n : \text{span}\{e_i : 1 \leq i \leq d_n\} \longrightarrow \ell_\infty$$

defined as

$$i_n = \lim_{m \rightarrow \infty} (i_{n,n+1} \circ i_{n+1,n+2} \circ \cdots \circ i_{m-1,m}).$$

It has been shown by J. Bourgain and F. Delbaen [3] that $\|i_n\| \leq \lambda$ for all $n \in \mathbb{N}$. Therefore, if we denote by

$$F_n = i_n(\text{span}\{e_i : 1 \leq i \leq d_n\}) \subseteq \ell_\infty,$$

the space of Bourgain-Delbaen $X_{a,b}$ is defined as the closure in ℓ_∞ of the countable union

$$\bigcup_{n \in \mathbb{N}} F_n.$$

By construction, it follows that the Banach-Mazur distance $d(F_n, \ell_\infty^{d_n}) \leq \lambda$ for each $n \in \mathbb{N}$. Thus,

(α) $X_{a,b}$ is a \mathcal{L}_∞ -space.

Curiously, the space $X_{a,b}$ satisfies the following properties, which make the space very interesting:

- (β) $X_{a,b}$ does not contain any copy of ℓ_1 , which implies that $X_{a,b}^*$ is isomorphic to ℓ_1 ;
- (γ) $X_{a,b}$ has the Radon-Nikodym property;
- (δ) $X_{a,b}$ is reflexive saturated (i.e., every infinite dimensional closed subspace contains an infinite dimensional reflexive subspace).

Before to close this section, we would like to review a last peculiar property. Let α be the unique number such that

$$a^{\frac{1}{1-\alpha}} + b^{\frac{1}{1-\alpha}} = 1.$$

Then, in $X_{a,b}$ the following holds.

Proposition 2.1. For every sequence $(x_n)_n \subseteq X_{a,b}$ such that

- (i) $\|x_n\| = 1$ for all $n \in \mathbb{N}$,
- (ii) $x_n \longrightarrow 0$ weakly,

there exist $C > 0$ and a subsequence $(x_{k_n})_n$ such that

$$(1) \quad \|x_{k_1} + \cdots + x_{k_n}\| \geq Cn^\alpha, \quad \forall n \in \mathbb{N}.$$

3. OPERATORS ON THE SPACE $X_{a,b}$

Throughout this section we fix a, b such that

- (i) $0 < b < a < 1$,
- (ii) $a + b > 1$,
- (iii) $a^3 + b^3 = 1$.

Before to state the main result, we would like to recall a well known result. Since the proof is easy, we insert it for the sake of completeness.

Proposition 3.1. Let X be a Banach space with a Schauder basis, Y any Banach space and let $T : X \rightarrow Y$ be a bounded linear operator. If

$$(2) \quad \|T(x_n)\| \xrightarrow{n \rightarrow \infty} 0 \text{ for all bounded block basic sequence } (x_n)_n,$$

then T is compact.

Proof. Let us denote by $(x_n)_n$ a Schauder basis of X with basis constant K , and let us denote by

$$P_n : X \rightarrow X$$

be the n -th projection associate to the basis; i.e.,

$$P_n\left(\sum_{i=1}^{\infty} a_i x_i\right) = \sum_{i=1}^n a_i x_i.$$

We show that

$$\|T - T \circ P_n\| \xrightarrow{n \rightarrow \infty} 0;$$

i.e., T is uniform limit of finite rank operators.

Let us assume we are not in this situation, that means there exist $\delta > 0$, a strictly increasing sequence of natural number $(k_n)_n$ and a sequence of norm-one $(z_n)_n \subseteq X$ such that

$$\|(T - T \circ P_{k_n})(z_n)\| > \delta.$$

Since $z_1 = \lim_n P_n(z_1)$, we can find $s_1 > k_1$ such that

$$\|z_1 - P_{s_1}(z_1)\| < \frac{\delta}{2\|T\|}.$$

Let us define $y_1 = P_{s_1}(z_1) - P_{k_1}(z_1)$. Then

$$\begin{aligned} \|y_1\| &\leq 2K \text{ and} \\ \|T(y_1)\| &\geq \|(T - T \circ P_{k_1})(z_1)\| - \|(T - T \circ P_{s_1})(z_1)\| \\ &> \frac{\delta}{2} \end{aligned}$$

Let $k_{j_2} > s_1$. Similarly, we find $s_2 > k_{j_2}$ such that

$$\|z_{j_2} - P_{s_2}(z_{j_2})\| < \frac{\delta}{2\|T\|}.$$

Let us define $y_2 = P_{s_2}(z_{j_2}) - P_{k_{j_2}}(z_{j_2})$. Then

$$\begin{aligned} \|y_2\| &\leq 2K \text{ and} \\ \|T(y_2)\| &> \frac{\delta}{2} \end{aligned}$$

Iterating this process we find a bounded block basic sequence $(y_n)_n$ of Y such that $\|T(y_n)\| > \frac{\delta}{2}$, against the assumption (2). \square

Theorem 3.2. Every bounded linear operator from ℓ_2 to $X_{a,b}$ and from $X_{a,b}$ to ℓ_2 must be compact.

Proof. Since $X_{a,b}^*$ has the Schur property, it is obvious that any operator $X_{a,b}$ to ℓ_2 is compact.

Suppose there is a bounded linear operator

$$T : \ell_2 \longrightarrow X_{a,b}$$

which is not compact. By the previous proposition, there must exists a bounded block basic sequence $(x_n)_n \subseteq \ell_2$ such that

$$\|T(x_n)\| > \delta, \text{ for some } \delta > 0.$$

It is worth to recall that a basis in a Banach space is shrinking if and only if any bounded block basic sequence is weakly null. Since the basis in ℓ_2 is shrinking, from one hand we have that the sequence $(x_n)_n$ is weakly null (and so it is $(T(x_n))_n$). Therefore, the sequence

$$\left(\frac{T(x_n)}{\|T(x_n)\|} \right)_n \text{ is weakly null.}$$

By Proposition 2.1 and assumption (iii) above, unless to pass through a subsequence, there exists $C > 0$ such that

$$\left\| \frac{T(x_1)}{\|T(x_1)\|} + \cdots + \frac{T(x_n)}{\|T(x_n)\|} \right\| \geq Cn^{\frac{2}{3}},$$

On the other hand, since bounded block basic sequence in ℓ_2 are unconditionally basic sequence (and so it is $(T(x_n))_n$), it follows also that

$$\|T(x_1) + \cdots + T(x_n)\| \geq C'n^{\frac{2}{3}},$$

for some $C' > 0$. Therefore,

$$\begin{aligned} C'n^{\frac{2}{3}} &\leq \|T(x_1) + \cdots + T(x_n)\| \\ &\leq \|T\| \|x_1 + \cdots + x_n\|_{\ell_2} \\ &\leq \|T\| C'' n^{\frac{1}{2}}, \end{aligned}$$

where the last inequality follows from the fact that $(x_n)_n$ is a bounded block sequence in ℓ_2 . This would imply that

$$n^{\frac{1}{6}} \leq \|T\| \frac{C''}{C'}.$$

Namely a contradiction. This completes the proof. \square

We would like to finish this note with the following natural questions.

Question. (i) Does there exists a pair of Banach spaces (X, Y) such that every n -homogeneous polynomial from X to Y and from Y to X must be compact?

(ii) Let Y and Z be reflexive Banach spaces. Does, for every Banach space X , the triple (X, Y, Z) has the compact operators transitivity property?

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCES, UNIVERSITY OF CATANIA,
CATANIA, 95125, ITALY (EU)

E-mail address: `dpuglisi@dmi.unict.it`